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Math 451: Intro. to General Topology — Homework 2 — **Due: Feb 3, 23:59**

Definition. Let Σ be a countable set and let $\Sigma^{<\mathbb{N}} := \bigcup_{n\in\mathbb{N}} \Sigma^n$ denote the set of all finite words in Σ. For a finite word $u \in \Sigma^{<\mathbb{N}}$ and a finite or infinite word $v \in \Sigma^{<\mathbb{N}} \cup \Sigma^{\mathbb{N}}$, we

- denote by uv for the concatenation of u and v.
- write $u \preceq v$ if v extends u, i.e. $v = uv'$ for some $v' \in \Sigma^{<\mathbb{N}} \cup \Sigma^{\mathbb{N}}$.

Call a subset $T \subseteq \Sigma^{<\mathbb{N}}$ a **tree** on Σ if it is nonempty and closed downward under $\preceq$, i.e. $v \in T$ implies $u \in T$ for all $u \preceq v$. In particular, the empty word is always in T. For a tree T on Σ, denote by $[T]$ the set of infinite branches through T, i.e.

$$[T] := \left\{x \in \Sigma^{\mathbb{N}} : \forall n \in \mathbb{N}, x|_n \in T\right\}.$$

1. Let (X, d) be a metric space and $Y \subseteq X$. Show that a subset $U \subseteq Y$ is open relative to Y (i.e. in the subspace Y) if and only if $U = V \cap Y$ for some open $V \subseteq X$ open (in the space X).

2. Prove that every open subset U of $\mathbb{R}$ is a countable union of disjoint open intervals.

Hint: Every $x \in U$ is contained in a maximal open interval I contained in U (obtain I by taking the union of all open intervals J with $x \in J \subseteq U$). Then U is a union of maximal open intervals. These are pairwise disjoint, hence each contains a rational that others don't contain, so there are only countably many maximal intervals.

3. Let Σ be a countable nonempty set. Define $d : \Sigma^{\mathbb{N}} \times \Sigma^{\mathbb{N}} \to [0, \infty)$ by

$$d(x,y) := \begin{cases} 2^{-\iota(x,y)} & \text{if } x \neq y \\ 0 & \text{otherwise,} \end{cases}$$

where $\iota(x, y)$ is the first index $i \in \mathbb{N}$ such that $x(i) \neq y(i)$. Prove:

(a) d is an ultrametric on $\Sigma^{\mathbb{N}}$, i.e. it satisfies the first two metric axioms, as well as the following stronger inequality than the triangle inequality: for all $x, y, z \in \Sigma^{\mathbb{N}}$,

$$d(x,z) \leqslant \max\{d(x,y), d(y,z)\}.$$

(b) Prove that a subset $C \subseteq \Sigma^{\mathbb{N}}$ is closed if and only if $C = [T]$ for some tree $T \subseteq \Sigma^{<\mathbb{N}}$.

(c) A sequence $(x_n) \subseteq \Sigma^{\mathbb{N}}$ converges to $x \in \Sigma^{\mathbb{N}}$ if and only if for each index $i \in \mathbb{N}$, the sequence $(x_n(i))$ is eventually $x(i)$, i.e. $\forall^\infty n \in \mathbb{N}$, $x_n(i) = x(i)$.

4. Let (x_n) be a sequence in a metric space X and $x \in X$. Prove that $\lim_n x_n = x$ if and only if every subsequence $(x_{n_k})_{k\in\mathbb{N}}$ has a further subsequence $(x_{n_{k_\ell}})_{\ell\in\mathbb{N}}$ which converges to x.

5. Let (X) be a metric space and $Y \subseteq X$. Prove that the following are equivalent:

(1) Y is closed.

(2) $Y = \overline{Y}$.

(3) Y contains all its adherent points.

(4) For every convergent sequence of elements from Y, its limit is also in Y.

Remark: Your proofs should be one sentence per implication, just putting together what we have proved in class.

6. Let (X,d) be a metric space and $Y \subseteq X$.

(a) Prove that for each $x \in X$, $d(x,Y) = 0$ if and only if $x \in \overline{Y}$.

(b) For each $r > 0$, call the set $B_r(Y) := \{x \in X : d(Y,x) < r\}$ the **open r-ball** around Y. Prove that $B_r(Y)$ is an open set (regardless of what kind of set Y is).

(c) Prove that $\overline{Y} = \bigcap_{n \in \mathbb{N}^+} B_{1/n}(Y)$.

(d) Conclude that every closed set is G_δ (i.e. a countable intersection of open sets), and hence every open set is F_σ (i.e. a countable union of closed sets).

7. (a) Prove that in metric spaces contractive sequences are Cauchy.

(b) Let $(x_n) \subseteq \mathbb{R}$ be defined recursively by

$$x_0 := 0; \quad x_1 := 1;$$
$$x_{n+2} := \tfrac{1}{2}(x_n + x_{n+1}).$$

Prove that (x_n) is contractive and convergent, and find the limit.

8. Prove that for each nonempty set Σ, the space $\Sigma^{\mathbb{N}}$ with the usual metric is complete.

Hint: Look at the picture in Lecture 8 at the bottom of page 2.

9. Define a complete metric d on $\mathbb{N}$, such that the sets $B_n := \{n, n+1, n+2, \ldots\}$ are closed balls. Thus, we would have a decreasing sequence (B_n) of closed balls in a complete metric space with empty intersection. This would demonstrate the necessity of the vanishing diameter condition.

Hint: For completeness, it suffices to make sure that $d(n,m) \geqslant 1$ for all $n \neq m$. For each $n \in \mathbb{N}$, the numbers greater than n should be closer to n than those smaller than n.

10. [*Optional*] Let X be a set. A **pseudo-metric** on X is a function $d : X \times X \to [0,\infty)$ satisfying all axioms of metric except $d(x,y) = 0 \implies x = y$; thus it satisfies, for all $x,y,z \in X$,

(i) $d(x,x) = 0$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,z) \leqslant d(x,y) + d(y,z)$.

Define a binary relation $\sim$ on X by setting $x \sim y$ when $d(x,y) = 0$.

(a) Show that $\sim$ is an equivalence relation on X.

(b) Let $\widetilde{X}$ denote the quotient $X/\sim$, and define $\widetilde{d} : \widetilde{X} \times \widetilde{X} \to [0,\infty)$ by setting

$$\widetilde{d}(C_1, C_2) := d(x_1, x_2),$$

where x_i is an arbitrary representative of $[x_i]_\sim$ for $i = 1,2$. Prove that $\widetilde{d}$ is well-defined, i.e. does not depend on the choice of representatives x_1, X_2 of the $\sim$-classes C_1, C_2. Furthermore, prove that $\widetilde{d}$ is a (genuine) metric on $X/\sim$.

Remark: This shows that every pseudo-metric space can be upgraded to an actual metric space.